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# The Homotopy Type of the Group of Automorphisms of a UHF-Algebra

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The homotopy groups of the group of *-automorphisms of a uniformly hyperfinite (UHF) $C^*$-algebra is calculated. This is done both when the group is endowed with the topology of pointwise norm-convergence and the topology of uniform norm-convergence. © 1987 Academic Press, Inc.

## INTRODUCTION

Since their appearance in Glimm's paper [3] in 1960, the uniformly hyperfinite (UHF) $C^*$-algebras have played an important role in the study of $C^*$- and $W^*$-algebras. They are among the best understood $C^*$-algebras. It is therefore only natural that the group of *-automorphisms of a UHF-algebra has been studied intensively, both earlier and in recent years. However, the knowledge about these groups is far from being so complete as the knowledge of the algebras themselves.

In the present paper we shall calculate the homotopy groups of these automorphism-groups with respect to the two most important topologies. The work may therefore be considered as a study of the topology of these groups. However, the author's own motivation for seeking the homotopy groups comes from efforts to classify certain $C^*$-algebras with a primitive ideal spectrum homeomorphic to a sphere.

The results are as follows:

Let $\{p_i\}_{i=1}^{\infty}$ be a sequence of primes and consider the UHF-algebra

$$B = \bigotimes_{i=1}^{\infty} M_{p_i}.$$

For each prime $p$, let $f(p)$ denote the number of times $p$ occurs in the sequence $\{p_i\}_{i=1}^{\infty}$ (such that $f(p) \in \mathbb{N} \cup \{0\} \cup \{\infty\}$) and $P_F$ the primes $p$



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such that $f(p) \in \mathbb{N}$. Let $\mathrm{Aut}(B)$ (resp. $\mathrm{Aut}_u(B)$) be the group of *-automorphisms of $B$ endowed with the topology of pointwise norm-convergence (resp. the topology of uniform norm-convergence). Then

$$\pi_{2k-1}(\mathrm{Aut}(B)) \cong \bigtimes_{p \in P_F} \mathbb{Z}/p^{f(p)}\mathbb{Z}$$

and

$$\pi_{2k}(\mathrm{Aut}(B)) = 0, \qquad k \in \mathbb{N}.$$

For $\mathrm{Aut}_u(B)$ we find

$$\pi_{2k+1}(\mathrm{Aut}_u(B)) = K_0(B),$$
$$\pi_{2k}(\mathrm{Aut}_u(B)) = 0, \qquad k \in \mathbb{N},$$

and

$$\pi_1(\mathrm{Aut}_u(B)) = K_0(B)/\mathbb{Z}$$

when $K_0(B)$ is identified (using the trace) with a subgroup of the rationals.

Since $\pi_0(\mathrm{Aut}(B)) = 0$ by a result of Blackadar [1] and $\pi_0(\mathrm{Aut}_u(B))$ is the group of outer autormorphisms of $B$ by a result of Lance [6], we have a complete list of the homotopy groups.

## 1

Let $A$ be a $C^*$-algebra and $\mathrm{Aut}(A)$ the group of *-automorphisms of $A$. $\mathrm{Aut}(A)$ carries (at least) two different topologies making it into a topological group. The most important is *the topology of pointwise norm-convergence* based on the open sets

$$\{\alpha \in \mathrm{Aut}(A) \mid \|\alpha(a) - a\| < 1\}, \qquad a \in A.$$

When we consider this topology, we simply write $\mathrm{Aut}(A)$ for the corresponding topological group.

The other topology is the *uniform norm-topology* based on the open sets

$$\{\alpha \in \mathrm{Aut}(A) \mid \sup_{a \neq 0} \|a\|^{-1} \|\alpha(a) - a\| < \varepsilon\}, \qquad \varepsilon > 0.$$

The corresponding topological group will be denoted by $\mathrm{Aut}_u(A)$.

For every $k \in \mathbb{N}$, we fix a reference point $e$ on the $k$-sphere $S^k$. For any topological group $G$ with trivial element 1, *a k-loop in G* will be a continuous function

$$\alpha: S^k \to G$$

such that $\alpha_e = 1$.

The corresponding element in $\pi_k(G)$ will be denoted by $[\alpha]$.

Let $M_n$, $n \in \mathbb{N}$, denote the $n \times n$ complex matrices and $U(n)$ the unitary group in $M_n$. Generally, we let $U(A)$ denote the unitary group of a unital $C^*$-algebra $A$. Let

$$B = M_{n_1} \otimes M_{n_2} \otimes M_{n_3} \otimes \cdots = \bigotimes_{i=1}^{\infty} M_{n_i} \tag{1}$$

be an (infinite-dimensional) UHF-algebra [3]. Then $B$ admits a factorisation

$$B = M_{p_1} \otimes M_{p_2} \otimes M_{p_3} \otimes \cdots = \bigotimes_{i=1}^{\infty} M_{p_i}, \tag{2}$$

where $\{p_i | i \in \mathbb{N}\}$ is a sequence of prime numbers. Let $P$ denote all the prime numbers. For each $p \in P$, we define

$$f(p) = \#\{i \in \mathbb{N} | p = p_i\},$$

i.e., $f(p)$ is the number of times $p$ occurs in the factorisation (2). Hence $f(p) \in \mathbb{N} \cup \{0\} \cup \{\infty\}$.

Note that $f$ is the same map as the one used by Glimm in [3] to classify the UHF-algebras. Glimm's result is that two UHF-algebras $B_1$, $B_2$, with functions $f_i: P \to \mathbb{N} \cup \{0, \infty\}$, $i = 1, 2$, respectively, are $*$-isomorphic if and only if $f_1 = f_2$.

That is, the factorisation (2) is essentially unique for a given UHF-algebra.

In what follows, we shall use the symbol $\mathbb{1}$ for the unit in varying $C^*$-algebras. However, it will always be clear from the context which algebra is meant. In the same spirit the identity map on any space will be denoted by id.

First, we will calculate the homotopy groups of $\mathrm{Aut}(B)$. For this, it is necessary to distinguish between two subsets of $P$,

$$P_F = \{p \in P | f(p) \in \mathbb{N}\},$$

$$P_\infty = \{p \in P | f(p) = \infty\}.$$

Given a factorisation of $B$ as in (1), we let $B_m$ denote the finite-dimensional $C^*$-subalgebra

$$B_m = M_{n_1} \otimes M_{n_2} \otimes M_{n_3} \otimes \cdots \otimes M_{n_m} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots,$$

$m \in \mathbb{N}$.

Given a family $\beta^i \colon S^k \to U(n_i)$, $i \in \mathbb{N}$, of unitary $k$-loops, we can define a $k$-loop $\beta$ in $\mathrm{Aut}(B)$ by

$$\beta_s = \mathrm{Ad}\, \beta_s^1 \otimes \mathrm{Ad}\, \beta_s^2 \otimes \mathrm{Ad}\, \beta_s^3 \otimes \cdots, \qquad s \in S^k.$$

Such a loop will be called a *product type k-loop in* $\mathrm{Aut}(B)$.

We start with some known results wich will be used repeatedly in what follows.

LEMMA 1.1 (Handelman). *Let $\sigma_n \colon A_n \to A_{n+1}$, $n \in \mathbb{N}$, be a sequence of injective unital *-homomorphisms between unital $C^*$-algebras. For each $k \in \mathbb{N}$, we obtain induced homomorphisms: $\pi_k(U(A_n)) \to \pi_k(U(A_{n+1}))$, $n \in \mathbb{N}$. Let $A$ denote the direct limit $C^*$-algebra*

$$A = \varinjlim_n A_n.$$

*Then the obvious homomorphism*

$$\varinjlim_n \pi_k(U(A_n)) \to \pi_k(U(A))$$

*is an isomorphism for each $k \in \mathbb{N}$.*

*Proof.* This is [4, Proposition 4.4]. ∎

LEMMA 1.2. *Whenever $n, k \in \mathbb{N}$ and $2n - 1 \geqq k$, the homomorphism*

$$U(n) \ni u \to \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in U(n+1)$$

*induce an isomorphism $\pi_k(U(n)) \to \pi_k(U(n+1))$ and*

$$\pi_k(U(n)) \cong \begin{cases} \mathbb{Z} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even.} \end{cases}$$

*Proof.* This is folklore. See, e.g., [5]. ∎

LEMMA 1.3. *Assume that $k \in \mathbb{N}$ is odd. Let $\{m_1, m_2, \ldots, m_n\} \subseteq \mathbb{N}$. Assume that*

$$2m_1 m_2 \cdots m_n - 1 \geqq k.$$

*Let $\lambda_i\colon \pi_k(U(m_i)) \to \pi_k(U(m_1 m_2 \cdots m_n))$ be the homomorphism induced by*

$$U(m_i) \ni u \to \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes u \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}, \qquad i = 1, 2, \dots, n,$$

*and let $\varphi\colon \pi_k(U(m_1 m_2 \cdots m_n)) \to \mathbb{Z}$ be an isomorphism.*

*For each $i \in \{1, 2, \dots, n\}$ such that $2m_i - 1 \geqq k$ there exists an isomorphism*

$$\varphi_i\colon \pi_k(U(m_i)) \to \mathbb{Z}$$

*such that the following diagram commutes*

$$\begin{array}{ccc} \pi_k(U(m_i)) & \xrightarrow{\lambda_i} & \pi_k(U(m_1 m_2 \cdots m_n)) \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi} \\ \mathbb{Z} & \xrightarrow{z \to m_1 m_2 \cdots \hat{m}_i \cdots m_n z} & \mathbb{Z} \end{array}$$

*where $m_1 m_2 \cdots \hat{m}_i \cdots m_n$ is the product of the elements in $\{m_1, m_2, \dots, m_n\} \setminus \{m_i\}$.*

*Proof.* We identify $M_{m_1} \otimes M_{m_2} \otimes \cdots \otimes M_{m_n}$ and $M_{m_1 m_2 \cdots m_n}$. Fix $i \in \{1, 2, \dots, n\}$ with the property that

$$2m_i - 1 \geqq k.$$

We can choose a unitary $v \in M_{m_1 m_2 \cdots m_n}$ such that

$$v(\mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes a \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})\, v^* = \begin{pmatrix} a & & & 0 \\ & a & & \\ & & a & \\ & & & \ddots & \\ 0 & & & & a \end{pmatrix}$$

for all $a \in M_{m_i}$.

Let $\alpha^i$ be a $k$-loop in $U(m_i)$ such that $[\alpha^i]$ generates $\pi_k(U(m_i)) \cong \mathbb{Z}$. Let

$$\beta^i = \begin{pmatrix} \alpha^i & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}.$$

Then $\beta^i$ is a $k$-loop in $U(m_1 m_2 \cdots m_n)$ and $[\beta^i]$ generates $\pi_k(U(m_1 m_2 \cdots m_n))$ by Lemma 1.2.

Hence we can assume, by interchanging $\alpha^i$ with $(\alpha^i)^*$ if necessary, that

$$[\beta^i] = \varphi^{-1}(1).$$

Since $\pi_0(U(m_1 m_2 \cdots m_n)) = 0$, we have

$$\begin{aligned}
\lambda_i([\alpha^i]) &= [\mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \alpha^i \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}] \\
&= [v(\mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \alpha^i \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})\, v^*] \\
&= \left[\begin{pmatrix} \alpha^i & & & 0 \\ & \alpha^i & & \\ & & \ddots & \\ 0 & & & \alpha^i \end{pmatrix}\right] = m_1 m_2 \cdots \hat{m}_i \cdots m_n \varphi^{-1}(1).
\end{aligned}$$

Define $\varphi^i(z[\alpha^i]) = z$, $z \in \mathbb{Z}$. ∎

Corollary 1.4. *Let $k \in \mathbb{N}$ be odd. Let $m_1, m_2, \ldots, m_n$ be as in Lemma 1.3. Assume furthermore that these numbers are mutually prime and that $2m_i - 1 \geqq k$ for all $i = 1, 2, \ldots, n$. Then every $k$-loop $u\colon S^k \to U(m_1 m_2 \cdots m_n)$ is homotopic to a product type $k$-loop,*

$$u_1 \otimes u_2 \otimes \cdots \otimes u_n,$$

*where each $u_i\colon S^k \to U(m_i)$ is a $k$-loop.*

*Proof.* It suffices to note that

$$\sum_i m_1 m_2 \cdots \hat{m}_i \cdots m_n \mathbb{Z} = \mathbb{Z}. \quad ∎$$

Corollary 1.5. *Let $n, m, k \in \mathbb{N}$ such that $2m - 1 \geqq k$. Then the homomorphism $\pi_k(U(m)) \to \pi_k(U(mn))$ induced by*

$$U(m) \ni u \to u \otimes \mathbb{1} \in U(mn)$$

*is injective.*

*Proof.* If $k$ is even, $\pi_k(U(m)) = \pi_k(U(mn)) = 0$. If $k$ is odd, the conclusion follows from Lemma 1.3. ∎

Corollary 1.6. *Adopt the assumptions and the notation of Lemma 1.3 with $n = 3$. Assume furthermore that $m_1$ and $m_3$ are mutually prime and that $2m_1 - 1 \geqq k$, $2m_3 - 1 \geqq k$. For each $k$-loop $\alpha$ in $U(m_1)$ the following conditions are equivalent:*

(a) *there is a k-loop $\beta$ in $U(m_3)$ such that*

$$[\alpha \otimes 1 \otimes 1] = [1 \otimes 1 \otimes \beta] \text{ in } \pi_k(U(m_1 m_2 m_3)).$$

(b) $\varphi_1([\alpha]) \in m_1 \mathbb{Z}$.

*If* (a) *is satisfied,* $\varphi_3([\beta]) \in m_3 \mathbb{Z}$.

*Proof.* If (a) is satisfied, we have that

$$m_2 m_3 \varphi_1([\alpha]) = m_1 m_2 \varphi_3([\beta]).$$

Since $m_1, m_3$ are mutually prime, this implies that $\varphi_1([\alpha]) \in m_1 \mathbb{Z}$ (and that $\varphi_3([\beta]) \in m_3 \mathbb{Z}$).

Conversely, if $\varphi_1([\alpha]) = m_1 z$, $z \in \mathbb{Z}$, we choose a $k$-loop $\beta$ in $U(m_3)$ such that

$$\varphi_2([\beta]) = z m_3.$$

Then $\varphi \circ \lambda_1([\alpha]) = m_2 m_3 m_1 z = \varphi \circ \lambda_3([\beta])$, or

$$\lambda_1([\alpha]) = \lambda_3([\beta])$$

which is (a). ∎

COROLLARY 1.7. *Adopt the assumptions and the notation of Lemma* 1.3 *with $n=2$. Assume that $m_1, m_2$ are mutually prime and that $2m_1 - 1 \geqq k$, $2m_2 - 1 \geqq k$. For each k-loop $\alpha$ in $U(m_1)$ the following conditions are equivalent*:

(a) *there is a k-loop $\beta$ in $U(m_2)$ such that $[\alpha \otimes 1] = [1 \otimes \beta]$ in $\pi_k(U(m_1 m_2))$*

(b) $\varphi_1([\alpha]) \in m_1 \mathbb{Z}$.

*Proof.* The proof is the same as for Corollary 1.6. ∎

LEMMA 1.8. *Let $k, n_1, n_2, m \in \mathbb{N}$ and assume that $n_1 | n_2$, i.e., $n_1$ divides $n_2$. Let $A = M_{n_1} \otimes M_m \otimes M_{n_2}$. If $\alpha$ is a k-loop in $U(n_1)$, then there is a k-loop $\beta$ in $U(n_2)$ such that*

$$[\alpha \otimes 1 \otimes 1] = [1 \otimes 1 \otimes \beta] \qquad \text{in } \pi_k(U(A)).$$

*Proof.* Let $d = n_2/n_1$ and let $\Pi\colon M_{n_2} \to M_{n_1} \otimes M_d$ be a *-isomorphism. Consider the *-isomorphism

$$\psi\colon A \to M_{n_1} \otimes M_m \otimes M_{n_1} \otimes M_d = C$$

given on simple tensors as

$$\psi(a\otimes b\otimes c)=a\otimes b\otimes \Pi(c).$$

Let $\varphi$ be the *-isomorphism of $C$ which exchange the first and the third factor of simple tensors. Then $\varphi=\operatorname{Ad} V$ for some unitary $V$ in $C$.

Since $\pi_0(U(C))=0$, we can find a curve

$$g\colon\ [0,1]\to U(C)$$

such that $g(0)=\mathbb{1}$, $g(1)=V$. Define $F(t,s)=\operatorname{Ad} g(t)(\alpha_s\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1})$, $t\in[0,1]$, $s\in S^k$. Then $F$ is a homotopy in $U(C)$ connecting the two $k$-loops $\alpha\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}$ and $\mathbb{1}\otimes\mathbb{1}\otimes\alpha\otimes\mathbb{1}$. Translating this homotopy back into $U(A)$ by $\psi^{-1}$ gives the desired conclusion. ∎

LEMMA 1.9. *Let $A$ be a unital $C^*$-algebra and let $\{e_{ij}\}_{i,j=1}^n$, $\{f_{ij}\}_{i,j=1}^n$ be two systems of matrix units in $A$ such that $\sum_i e_{ii}=\sum_i f_{ii}=\mathbb{1}$. Assume $\|e_{11}-f_{11}\|=t<1$ and let $g$ be a continuous function on $[0,1]$ such that $g$ is 0 in a neighbourhood of 0 and*

$$g(x)=\frac{1}{x}\qquad \text{if}\quad x\geqq 1-t.$$

*Then $u=\sum_{i=1}^n e_{i1}[g(e_{11}f_{11}e_{11})]^{1/2}f_{1i}$ is a unitary in $A$ such that $e_{ij}=uf_{ij}u^*$, $i,j=1,2,\ldots,n$.*

*Proof.* By the proof of [3, Lemma 1.8], $e_{ii}[g(e_{11}f_{11}e_{11})]^{1/2}f_{ii}$ is a partial isometry from $f_{ii}$ to $e_{ii}$ for each $i$. And as noted by Bratteli in the proof of [2, Lemma 2.3], $u$ is a unitary with the stated property. ∎

LEMMA 1.10. *Let $B=M_{n_1}\otimes M_{n_2}\otimes M_{n_3}\otimes\cdots=\bigotimes_{i=1}^\infty M_{n_i}$ be a* UHF-*algebra. Let $\alpha\colon S^k\to\operatorname{Aut}(B)$ be a $k$-loop in $\operatorname{Aut}(B)$. For every $m\in\mathbb{N}$, there exists a $k$-loop*

$$U^m\colon S^k\to U(B)$$

*such that $\alpha_s|_{B_m}=\operatorname{Ad} U_s^m|_{B_m}$ for all $s\in S^k$.*

*Proof.* Fix $m$ and let $B_m=A$. Consider $\alpha$ as a continuous map $\alpha\colon[0,1]^k\to\operatorname{Aut}(B)$ which is trivial on the boundary. Let $\{e_{ij}\}_{i,j}$ be a set of matrix units in $A$ with $\sum_i e_{ii}=\mathbb{1}$. Choose $N\in\mathbb{N}$ such that $|t_1-t_2|\leqq 1/N$ implies that

$$\sup\{\|(\alpha_{t_1,x}-\alpha_{t_2,x})|_A\|\,|x\in[0,1]^{k-1}\}<\tfrac{1}{2}.$$

Let $g$ be a function on $[0,1]$ as in Lemma 1.9 (with $t=\frac{1}{2}$). Define $W_{t,x}^j=\sum_i\alpha_{t,x}(e_{i1})[g(\alpha_{t,x}(e_{11})\,\alpha_{j/N,x}(e_{11})\,\alpha_{t,x}(e_{11}))]^{1/2}\quad\alpha_{j/N,x}(e_{1i})$ for $t\in$

$[j/N, (j+1)/N]$, $x \in [0, 1]^{k-1}$, $j = 0, 1, 2, \ldots, N-1$. Then $W^j_{t,x}$ depends continuously on $(t, x)$ for all $j$, and by Lemma 1.9

$$W^m_{t,x} \alpha_{m/N,x}(e_{ij})(W^m_{t,x})^* = \alpha_{t,x}(e_{ij})$$

for all $i, j$, $t \in [m/N, (m+1)/N]$, $x \in [0, 1]^{k-1}$, $m = 0, 1, 2, \ldots, N-1$. Also $W^j_{j/N,x} = \mathbb{1}$ for all $x \in [0, 1]^{k-1}$ and $W^j_{t,x} = \mathbb{1}$ whenever $x$ lies on the boundary of $[0, 1]^{k-1}$. Define

$$V_{t,x} = \begin{cases} W^0_{t,x}, & t \in \left[0, \frac{1}{N}\right], \quad x \in [0, 1]^{k-1} \\ W^j_{t,x} W^{j-1}_{j/N,x} W^{j-2}_{(j-1)/N,x} \cdots W^0_{1/N,x}, & t \in \left[\frac{j}{N}, \frac{j+1}{N}\right], \quad x \in [0, 1]^{k-1}. \end{cases}$$

Then $V: [0, 1]^k \to U(B)$ is continuous, $V_{0,x} = \mathbb{1}$ for all $x \in [0, 1]^{k-1}$ and $V_{t,x} = \mathbb{1}$ for all $t \in [0, 1]$ when $x$ is on the boundary of $[0, 1]^{k-1}$. Also, $\alpha_{t,x}|_A = \operatorname{Ad} V_{t,x}|_A$ for all $(t, x) \in [0, 1]^k$. Hence $[0, 1]^{k-1} \ni x \to V_{1,x}$ is a $(k-1)$-loop in $U(B \cap A')$.

*Claim.* $[V_1] = 0$ in $\pi_{k-1}(U(B \cap A'))$.

The claim is automatic if $k-1$ is even, since $\pi_{k-1}(U(B \cap A')) = 0$ in this case by Lemmas 1.1 and 1.2. So assume that $k-1$ is odd. Let $W$ be the $(k-1)$-loop in

$$C = M_{n_{m+1}} \otimes M_{n_{m+2}} \otimes M_{n_{m+3}} \otimes \cdots,$$

such that $V_1 = \mathbb{1} \otimes W$.

By Lemma 1.1 there exist $l > m$ and a $(k-1)$-loop $\beta$ in $M_{n_{m+1} n_{m+2} \cdots n_l}$ such that

$$[W] = [\beta \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots] \quad \text{in} \quad \pi_{k-1}(U(C)).$$

Then $[\mathbb{1} \otimes \beta \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots] = [\mathbb{1} \otimes W] = [V_1]$ in $\pi_{k-1}(U(B))$. But $[V_1] = 0$ in $\pi_{k-1}(U(B))$ since the map

$$[0, 1] \times [0, 1]^{k-1} \ni (t, x) \to V_{t,x}$$

defines a homotopy in $U(B)$ connection $V_1$ to the trivial $(k-1)$-loop in $U(B)$. So if we assume, as we may, that

$$2n_{m+1} n_{m+2} \cdots n_l - 1 \geqq k-1,$$

it follows from Lemma 1.1 and Corollary 1.5 that

$$[\beta] = 0 \quad \text{in} \quad \pi_{k-1}(U(n_{m+1} n_{m+2} \cdots n_l)).$$

Thus $[V_1] = [\mathbb{1} \otimes \beta \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots] = 0$ in $\pi_{k-1}(U(B \cap A'))$ as claimed.

It follows that there is a continuous map

$$F\colon [0, 1] \times [0, 1]^{k-1} \to U(B \cap A')$$

such that $F(t, \cdot)$ is $(k-1)$-loop in $U(B \cap A')$ for all $t \in [0, 1]$ while

$$F(0, x) = \mathbb{1}, \qquad F(1, x) = V^*_{1,x}, \quad x \in [0, 1]^{k-1}.$$

Define $U_{t,x} = V_{t,x} F_{t,x}$, $(t, x) \in [0, 1] \times [0, 1]^{k-1}$. Then $U$ is a $k$-loop in $U(B)$ such that

$$\alpha_s|_A = \operatorname{Ad} U_s|_A \qquad \text{for all } s \in [0, 1]^k. \quad \blacksquare$$

LEMMA 1.11. *Let $B = \bigotimes_{i=1}^{\infty} M_{n_i}$ be a* UHF*-algebra and $\alpha^1$ a $k$-loop in $U(n_1)$. Assume that there is a strictly increasing sequence $\{m(i)\}_{i=1}^{\infty} \subseteq \mathbb{N}$ such that $m(1) = 1$ and $k$-loops $\alpha^i$ in $U(n_{m(i)})$ such that*

$$[\alpha^i \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}] = [\mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \alpha^{i+1}]$$

*in $\pi_k(U(n_{m(i)} n_{m(i)+1} \cdots n_{m(i+1)}))$, $i \in \mathbb{N}$. Then*

$$[\operatorname{Ad} \alpha^1 \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \cdots] = 0 \qquad \text{in } \pi_k(\operatorname{Aut}(B)).$$

*Proof.* Let $0 = \delta_0 < \delta_1 < \delta_2 < \cdots$, be a strictly increasing sequence with $\lim_n \delta_n = 1$. Let $F_i\colon [\delta_i, \delta_{i+1}] \times S^k \to \operatorname{Aut}(B)$ be continuous maps satisfying the following requirements:

(a) $F_i(\delta_i, s) = \mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \operatorname{Ad} \alpha_s^{i+1} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \ldots$, $s \in S^k$,

(b) $F_i(\delta_{i+1}, s) = \mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \operatorname{Ad} \alpha_s^{i+2} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \ldots$, $s \in S^k$,

(c) $F_i(t, e) = \mathrm{id}$, $t \in [\delta_i, \delta_{i+1}]$, $i = 0, 1, 2, \ldots$,

(d) $F_i(t, s)(a) = a$ for all $B_{m(i+1)-1}$, $t \in [\delta_i, \delta_{i+1}]$, $s \in S^k$, $i = 1, 2, 3, \ldots$.

This family of maps exists by assumption. Then define

$$F\colon [0, 1] \times S^k \to \operatorname{Aut}(B)$$

by

$$F(t, s) = \begin{cases} F_i(t, s), & t \in [\delta_i, \delta_{i+1}], \quad s \in S^k \\ \mathrm{id}, & t = 1, \quad s \in S^k. \end{cases}$$

If $a \in B_d$ and $m(i+1) > d$, then

$$F(t, s)(a) = a \qquad \text{for all } t \geqq \delta_i,\ s \in S^k.$$

This is an immediate consequence of (d). This observation together with (a), (b), and (c) show that $F$ is a homotopy connecting $\operatorname{Ad} \alpha^1 \otimes \mathrm{id} \otimes \mathrm{id} \otimes \cdots$, to the trivial loop in $\operatorname{Aut}(B)$. $\blacksquare$

LEMMA 1.12. *Let* $B = \bigotimes_{i=1}^{\infty} M_{p_i}$ *be a* UHF-*algebra and assume that* $\# P_F = \infty$. *Let* $\{N_i | i \in \mathbb{N}\}$ *be a partition of* $P_F$ *into disjoint subsets such that*

$$\prod_{p \in N_i} p^{f(p)} = q_i < \infty, \qquad i \in \mathbb{N},$$

*satisfy that* $2q_i - 1 \geq k$. *Then* $B$ *is* *-*isomorphic to an infinite tensor-product*

$$A = M_{q_1} \otimes M_{l_1} \otimes M_{q_2} \otimes M_{l_2} \otimes M_{q_3} \otimes M_{l_3} \otimes \cdots,$$

*such that the prime decomposition of each* $l_i$ *only contains powers of elements from* $P_\infty$ *and* $2l_i - 1 \geq k$ *or* $l_1 = l_2 = l_3 = \cdots = 1$. *Also, we may assume that*

$$l_1 l_2 l_3 \cdots l_n | l_{n+1} \qquad \textit{for all } n \in \mathbb{N}.$$

*Every element of* $\pi_k(\mathrm{Aut}(A))$ *is represented by a product type* $k$-*loop* $\mathrm{Ad}\, \beta_1 \otimes \mathrm{id} \otimes \mathrm{Ad}\, \beta_2 \otimes \mathrm{id} \otimes \mathrm{Ad}\, \beta_3 \otimes \mathrm{id} \otimes \ldots$, *where* $\beta_i \colon S^k \to U(q_i)$ *are* $k$-*loops*.

*Proof.* That $B$ is *-isomorphic to an $A$ of the given form follows from Glimm's classification [3]. The case $l_1 = l_2 = \cdots = 1$ occurs if $P_\infty = \emptyset$.

Let $\alpha \colon S^k \to \mathrm{Aut}(A)$ be a $k$-loop. By Lemma 1.10 we have $k$-loops $U^n \colon S^k \to U(A)$ such that

$$\alpha|_{A_n} = \mathrm{Ad}\, U^n|_{A_n}, \qquad n \in \mathbb{N}.$$

We claim that every $k$-loop in $\mathrm{Aut}(A)$ of the form $\mathrm{Ad}(U)$, where $U$ is a $k$-loop in $U(A)$, is homotopic to a product $k$-loop in $\mathrm{Aut}(A)$ of the form

$$\mathrm{Ad}\, \alpha^1 \otimes \mathrm{id} \otimes \mathrm{Ad}\, \alpha^2 \otimes \mathrm{id} \otimes \mathrm{Ad}\, \alpha^3 \otimes \mathrm{id} \otimes \ldots,$$

where $\alpha^i$ is a $k$-loop in $U(q_i)$, $i \in \mathbb{N}$.

In order to prove this, we can assume, by Lemma 1.1, that $U$ is a $k$-loop in $U(q_1 l_1 q_2 l_2 \cdots l_n)$ for some $n \in \mathbb{N}$. Let

$$\sigma \colon M_{q_1} \otimes M_{l_1} \otimes M_{q_2} \otimes \cdots \otimes M_{l_n} \to M_{q_1} \otimes M_{q_2} \otimes \cdots \otimes M_{q_n} \otimes M_{l_1 l_2 \cdots l_n} = C$$

be the obvious *-isomorphism, and let

$$\tilde{\sigma} \colon A \to C \otimes M_{q_{n+1}} \otimes M_{l_{n+1}} \otimes M_{q_{n+2}} \otimes \cdots = C'$$

be the corresponding *-isomorphism. Then $\tilde{\sigma}(U) = \sigma(U) \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots$ is homotopic to a $k$-loop of the form

$$\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n \otimes \gamma_n \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots,$$

where $\alpha_i$ is a $k$-loop in $U(q_i)$ and $\gamma_n$ a $k$-loop in $U(l_1 l_2 \cdots l_n)$. This follows from Corollary 1.4

Applying first Lemma 1.8 inductively and then Lemma 1.11, we conclude that $\operatorname{Ad}\tilde{\sigma}(U)$ is homotopic in $\operatorname{Aut}(C')$ to the $k$-loop $\operatorname{Ad}\alpha_1 \otimes \operatorname{Ad}\alpha_2 \otimes \cdots \otimes \operatorname{Ad}\alpha_n \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \ldots$.

Hence $\operatorname{Ad} U$ is homotopic in $\operatorname{Aut}(A)$ to the $k$-loop

$$\operatorname{Ad}\alpha_1 \otimes \mathrm{id} \otimes \operatorname{Ad}\alpha_2 \otimes \mathrm{id} \otimes \operatorname{Ad}\alpha_3 \otimes \mathrm{id} \otimes \cdots.$$

This proves the claim (and even more).

Let $V_n = U_{2n}$, $n \in \mathbb{N}$. Then $V_n^* V_{n+1}$ is a $k$-loop in $U(A \cap A'_{2n})$, $n \in \mathbb{N}$. Applying the preceeding claim, we obtain continuous maps

$$F_1\colon \quad [0,1] \times S^k \to \operatorname{Aut}(A)$$

and

$$G_n\colon \quad [0,1] \times S^k \to \operatorname{Aut}(A)$$

such that

$$\begin{aligned}
&F_1(0,s) = \operatorname{Ad} V_1(s),\\
&F_1(1,s) = \operatorname{Ad}\beta_0^1(s) \otimes \mathrm{id} \otimes \operatorname{Ad}\beta_0^2(s) \otimes \mathrm{id} \otimes \operatorname{Ad}\beta_0^3(s) \otimes \cdots,\\
&F_1(t,e) = \mathrm{id}, \qquad s \in S^k,\ t \in [0,1],\\
&G_n(0,s) = \operatorname{Ad} V_n^*(s)\, V_{n+1}(s), \qquad s \in S^k,\\
&G_n(1,s) = \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \operatorname{Ad}\beta_n^{n+1}(s) \otimes \mathrm{id} \otimes \operatorname{Ad}\beta_n^{n+2}(s) \otimes \mathrm{id} \otimes \cdots,\\
&G_n(t,e) = \mathrm{id},
\end{aligned}$$

and $G_n(t,s)|_{A_{2n}} = \mathrm{id}|_{A_{2n}}$, $s \in S^k$, $t \in [0,1]$, $n = 1, 2, 3, \ldots$. Here $\beta_j^i$ are $k$-loops in $U(q_i)$ for all $i, j \in \mathbb{N}$. Let

$$\lambda_\infty(s) = \operatorname{Ad}\beta_0^1(s) \otimes \mathrm{id} \otimes \operatorname{Ad}\beta_0^2(s)\,\beta_1^2(s) \otimes \mathrm{id} \otimes \operatorname{Ad}\beta_0^3(s)\,\beta_1^3(s)\,\beta_2^3(s)\ldots, \quad s \in S^k.$$

Let $\lambda_1(s) = F_1(1,s)$, $\lambda_n(s) = G_{n-1}(1,s)$, $n \geqq 2$, $s \in S^k$. Choose an increasing sequence $0 = \delta_0 < \delta_1 < \delta_2 < \ldots$, converging to 1.

Using the homotopies $F_1, G_1, G_2, G_3, \ldots$, we construct continuous maps

$$H_n\colon [\delta_{n-1}, \delta_n] \times S^k \to \operatorname{Aut}(A), \qquad n \in \mathbb{N},$$

such that

$$\begin{aligned}
&H_1(0,s) = \lambda_\infty(s) \qquad (= \lambda_1(s)\,\lambda_1(s)^{-1}\lambda_\infty(s)),\\
&H_1(\delta_1,s) = \operatorname{Ad} V_1(s)\,\lambda_1(s)^{-1}\lambda_\infty(s), \qquad s \in S^k,\\
&H_1(t,e) = \mathrm{id}, \qquad t \in [0,\delta_1],
\end{aligned}$$

and

$$\begin{aligned} H_n(\delta_{n-1}, s) &= \mathrm{Ad}\, V_{n-1}(s)\, \lambda_{n-1}^{-1}(s)\, \lambda_{n-2}^{-1}(s) \cdots \lambda_1^{-1}(s)\, \lambda_\infty(s), \\ H_n(\delta_n, s) &= \mathrm{Ad}\, V_n(s)\, \lambda_n^{-1}(s)\, \lambda_{n-1}^{-1}(s) \cdots \lambda_1^{-1}(s)\, \lambda_\infty(s) \\ H_n(t, e) &= \mathrm{id}, \qquad t \in [\delta_{n-1}, \delta_n], s \in S^k. \end{aligned}$$

Observe that

$$H_n(t, s)|_{A_{2(n-1)}} = \mathrm{Ad}\, V_{n-1}(s)|_{A_{2(n-1)}} = \alpha_s|_{A_{2(n-1)}}$$

for $t \in [\delta_{n-1}, \delta_n]$, $s \in S^k$, $n \geqq 2$. Define

$$H(t, s) = \begin{cases} H_n(t, s), & t \in [\delta_{n-1}, \delta_n] \\ \alpha_s, & t = 1, \quad s \in S^k. \end{cases}$$

Then $H$ is a homotopy connecting $\alpha$ and $\lambda_\infty$. ∎

LEMMA 1.13. *Let $B = \bigotimes_{i=1}^\infty M_{p_i}$ and assume that $\# P_F < \infty$. Then $B$ is *-isomorphic to*

$$A = M_n \otimes M_{q_1} \otimes M_{q_2} \otimes M_{q_3} \otimes \cdots,$$

*where $n = \prod_{p \in P_F} p^{f(p)}$ and $\{q_i | i \in \mathbb{N}\} = P_\infty$.*

*For every $k$-loop $\alpha: S^k \to \mathrm{Aut}(A)$ there exist $m \in \mathbb{N}$ and a $k$-loop $\beta$ in $U(A_m)$ such that*

$$[\alpha] = [\mathrm{Ad}\, \beta] \qquad \textit{in} \quad \pi_k(\mathrm{Aut}(A)).$$

*Proof.* The *-isomorphism part of the statement follows from Glimm's classification [3]. According to Lemma 1.10 there exist $k$-loops $\beta_m$ in $U(A)$ such that

$$\alpha|_{A_m} = \mathrm{Ad}\, \beta_m|_{A_m} \qquad \text{for all } m \in \mathbb{N}.$$

For each $m$, $\beta_{m+1}^* \beta_m$ is a $k$-loop in $U(A \cap A_m')$. By Lemma 1.1 there is a $k$-loop $V$ in $U(A_i \cap A_m')$ for some $i \geqq m$ such that

$$[V] = [\beta_{m+1}^* \beta_m] \qquad \text{in} \quad \pi_k(U(A \cap A_m')), \quad m \in \mathbb{N}.$$

Applying Lemma 1.8 inductively, and then Lemma 1.11, this gives that $[\mathrm{Ad}\, \beta_{m+1}^* \beta_m] = [\mathrm{Ad}\, V] = 0$ in $\pi_k(\mathrm{Aut}(A \cap A_m'))$ for all $m \in \mathbb{N}$.

Let $0 = \delta_0 < \delta_1 < \delta_2 < \cdots$, be an increasing sequence with $\lim_n \delta_n = 1$. Then we have continuous maps

$$G_i: [\delta_{i-1}, \delta_i] \times S^k \to \mathrm{Aut}(A)$$

such that

$$\begin{aligned} G_i(\delta_{i-1}, s) &= \mathrm{Ad}(\beta^*_{i+1}(s)\, \beta_i(s)) \\ G_i(\delta_i, s) &= \mathrm{id} \\ G_i(t, e) &= \mathrm{id} \\ G_i(t, s)|_{A_i} &= \mathrm{id}|_{A_i}, \qquad t \in [\delta_{i-1}, \delta_i], \quad s \in S^k, \quad i \in \mathbb{N}. \end{aligned}$$

Let

$$F_i(t, s) = \mathrm{Ad}\, \beta_{i+1}(s)\, G_i(t, s),$$
$$t \in [\delta_{i-1}, \delta_i], \quad s \in S^k, \quad i = 1, 2, 3, ...,$$

and define

$$F(t, s) = \begin{cases} F_i(t, s), & t \in [\delta_{i-1}, \delta_i] \\ \alpha_s, & t = 1, \quad s \in S^k. \end{cases}$$

If $a \in A_i$, $t \geqq \delta_{i-1}$, we find that

$$F(t, s)(a) = \alpha_s(a) \qquad \text{for all } s \in S^k.$$

It is then straightforward to check that $F$ is a homotopy connecting $\mathrm{Ad}\, \beta_1$ to $\alpha$.

In order to conclude the proof, it suffices to refer to Lemma 1.1. ∎

LEMMA 1.14. *Let $B = \bigotimes_{i=n}^{\infty} M_{n_i}$. Let $\alpha$ be a k-loop in $U(n_1 n_2 \cdots n_m)$ and let $\beta$ be the k-loop* $\mathrm{Ad}\, \alpha \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \cdots$, *in* $\mathrm{Aut}(B)$. *Assume that there is a continuous map*

$$G\colon [0, 1] \times S^k \to \mathrm{Aut}(B)$$

*such that*

$$\begin{aligned} G(0, s) &= \beta_s, & s \in S^k, \\ G(t, e) &= \mathrm{id}, & t \in [0, 1], \end{aligned}$$

*and*

$$G(1, s)|_{B_m} = \mathrm{id}|_{B_m}, \qquad s \in S^k.$$

*Then there exist $l > m$ and a k-loop $\lambda$ in $U(n_{m+1} n_{m+2} \cdots n_l)$ such that*

$$[\alpha \otimes \mathbb{1}] = [\mathbb{1} \otimes \lambda]$$

*in $\pi_k(U(n_1 n_2 \cdots n_l))$.*

*Proof.* Choose $N \in \mathbb{N}$ such that $|t_1 - t_2| \leqslant 1/N$ implies that

$$\sup_{s \in S^k} \|(G(t_1, s) - G(t_2, s))|_{B_m}\| < \tfrac{1}{2}.$$

Let $\{e_{ij}\}$ be a system of matrix units in $B_m$ with $\sum_i e_{ii} = \mathbb{1}$, and let $g$ be a continuous function as in Lemma 1.9 (with $t = \frac{1}{2}$). Define

$$F_j(t, s) = \sum_i G(t, s)(e_{i1}) \left[ g \left( G(t, s)(e_{11})\, G\left(\frac{j}{N}, s\right)(e_{11}) \right.\right.$$
$$\left.\left. \times G(t, s)(e_{11}) \right) \right]^{1/2} G\left(\frac{j}{N}, s\right)(e_{1i})$$

for $t \in [j/N, (j+1)/N]$, $s \in S^k$, $j = 0, 1, 2,..., N-1$.

Then $G(t, s)(a) = \operatorname{Ad} F_j(t, s)(G(j/N, s)(a))$ for all $t \in [j/N, (j+1)/N]$, $s \in S^k$, $a \in B_m$, $j = 0, 1, 2,..., N-1$ by Lemma 1.9. $F_j$ is continuous and $F_j(t, e) = \mathbb{1} = F_j(j/N, s)$ for all $t \in [j/N, (j+1)/N]$, $s \in S^k$, $j = 0, 1, 2,..., N-1$. We define

$$F(t, s) = F_j(t, s) \prod_{i=0}^{j-1} F_j\left(\frac{j+1}{N}, s\right)(\alpha_s \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots),$$

$t \in [j/N, (j+1)/N]$, $s \in S^k$, $j = 0, 1, 2,..., N-1$. Then $F$ is a homotopy in $U(B)$ and

$$\operatorname{Ad} F(t, s)|_{B_m} = G(t, s)|_{B_m}$$

for all $t \in [0, 1]$, $s \in S^k$. Especially, $F(1, s) \in B \cap B'_m$ for all $s \in S^k$. Thus $F$ is a homotopy in $U(B)$ connecting the $k$-loop $\alpha \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots$, in $U(B)$ to a $k$-loop in $U(B \cap B'_m)$.

By Lemma 1.1 there exist $l > m$ and a $k$-loop $\lambda$ in $U(n_{m+1} n_{m+2} \cdots n_l)$ such that the $k$-loop $F(1, \cdot)$ is homotopic in $U(B \cap B'_m)$ to the $k$-loop $\mathbb{1} \otimes \lambda \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots$. Hence $\alpha \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots$, is homotopic in $U(B)$ to $\mathbb{1} \otimes \lambda \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots$. We can assume that $l$ is so large that

$$2n_1 n_2 \cdots n_l - 1 \geqslant k.$$

Then Lemma 1.1 and Corollary 1.5 in combination yield that

$$[\alpha \otimes \mathbb{1}] = [\mathbb{1} \otimes \lambda]$$

in $\pi_k(U(n_1, n_2 \cdots n_l))$. ∎

THEOREM 1.15. *Let* $B = \bigotimes_{i=1}^{\infty} M_{p_i}$ *be an infinite-dimensional* UHF-*algebra. Then*

$$\pi_{2k}(\mathrm{Aut}(B)) = 0, \qquad k = 0, 1, 2, \ldots,$$

*and*

$$\pi_{2k+1}(\mathrm{Aut}(B)) = \bigtimes_{p \in P_F} \mathbb{Z}_{p^{f(p)}}, \qquad k = 0, 1, 2, \ldots.$$

*Proof.* $\pi_0(\mathrm{Aut}(B)) = 0$ by [1, Theorem 2.3]. That $\pi_{2k}(\mathrm{Aut}(B)) = 0$ for $k = 1, 2, \ldots$, follows immediately from Lemmas 1.12, 1.13, and 1.2. Hence we turn to the odd case.

First, assume that $\# P_F < \infty$. We may assume that

$$B = M_n \otimes M_{q_1} \otimes M_{q_2} \otimes M_{q_3} \otimes \cdots,$$

where $n = \prod_{p \in P_F} p^{f(p)}$ and $\{q_i \mid i \in \mathbb{N}\} = P_\infty$. Choose $m_0 \in \mathbb{N}$ such that $2q_1 q_2 \cdots q_{m_0} - 1 \geq 2k+1$. Let $\gamma_m \colon \pi_{2k+1}(U(nq_1 \cdots q_m)) \to \pi_{2k+1}(\mathrm{Aut}(B))$ be induced by

$$U(nq_1 \cdots q_m) \ni u \to \mathrm{Ad}\, u \otimes \mathrm{id} \otimes \mathrm{id} \otimes \cdots, \qquad m \in \mathbb{N}.$$

Applying Lemma 1.3 we get an infinite commutative diagram

$$\begin{tikzcd}
 & \pi_{2k+1}(\mathrm{Aut}(B)) & \\
\pi_{2k+1}(U(nq_1 q_2 \cdots q_{m_0})) \arrow[r, "\lambda_{m_0}"] \arrow[ur, "\gamma_{m_0}"] \arrow[d, "\psi_{m_0}"] & \pi_{2k+1}(U(nq_1 q_2 \cdots q_{m_0+1})) \arrow[r, "\lambda_{m_0+1}"] \arrow[u, "\gamma_{m_0+1}"] \arrow[d, "\psi_{m_0+1}"] & \cdots \arrow[ul, "\gamma_{m_0+2}"'] \\
\mathbb{Z} \arrow[r, "z \to q_{m_0+1} z"] & \mathbb{Z} \arrow[r, "z \to q_{m_0+2} z"] & \cdots,
\end{tikzcd}$$

where each $\psi_m$ is an isomorphism.

*Claim.* $\ker \gamma_m = \psi_m^{-1}(n\mathbb{Z})$, $m \geq m_0$.

To see this, first assume that $\alpha$ is a $(2k+1)$-loop in $U(nq_1 q_2 \cdots q_m)$ and that $\psi_m([\alpha]) \in n\mathbb{Z}$. By Lemma 1.3 this implies that there is a $(2k+1)$-loop $\beta$ in $U(q_1 q_2 \cdots q_m)$ such that

$$[\alpha] = [\mathbb{1} \otimes \beta] \qquad \text{in} \quad \pi_{2k+1}(U(nq_1 q_2 \cdots q_m)).$$

Since $\{q_1, q_2, \ldots, q_m\} \subseteq P_\infty$, Lemma 1.8 and Lemma 1.11 together give that $[\mathrm{id} \otimes \mathrm{Ad}\, \beta \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \cdots] = 0$ in $\pi_{2k+1}(\mathrm{Aut}(B))$. Hence

$[\alpha] \in \operatorname{Ker} \gamma_m$. Conversely, if $\gamma_m([\alpha])=0$ Lemma 1.14 gives $l>m$ and a $(2k+1)$-loop $\beta$ in $U(q_{m+1}q_{m+2}\cdots q_l)$ such that

$$[\alpha \otimes \mathbb{1}] = [\mathbb{1} \otimes \beta] \qquad \text{in} \quad \pi_k(U(nq_1q_2\cdots q_l)).$$

Since we can assume that $2q_{m+1}q_{m+2}\cdots q_l - 1 \geqslant 2k+1$, Lemma 1.3 gives that

$$q_{m+1}q_{m+2}\cdots q_l \psi_m([\alpha]) \in nq_1q_2\cdots q_m\mathbb{Z}$$

from which we conclude that $\psi_m([\alpha]) \in n\mathbb{Z}$. Thus the claim is verified. Hence we obtain a commutative diagram

$$\begin{tikzcd}
\pi_{2k+1}(\mathrm{Aut}(B)) & & \\
\mathbb{Z}_n \arrow[u, "\tilde{\gamma}_{m_0}"] \arrow[r] & \mathbb{Z}_n \arrow[ul, "\tilde{\gamma}_{m_0+1}"'] \arrow[r] & \mathbb{Z}_n \arrow[ull, "\tilde{\gamma}_{m_0+2}"'] \arrow[r] & \cdots,
\end{tikzcd}$$

where the horizontal lines indicate the automorphisms of $\mathbb{Z}_n$ induced by

$$z \to q_m z, \qquad m \geqslant m_0,$$

and each $\tilde{\gamma}_m$ is injective. But

$$\bigcup_{m \geqslant m_0} \tilde{\gamma}_m(\mathbb{Z}_n) = \bigcup_{m \geqslant m_0} \gamma_m(\pi_{2k+1}(U(nq_1q_2\cdots q_m))) = \pi_{2k+1}(\mathrm{Aut}(B))$$

by Lemma 1.13. Thus each $\tilde{\gamma}_m$, $m \geqslant m_0$, is an isomorphism. Hence $\pi_{2k+1}(\mathrm{Aut}(B)) \simeq \mathbb{Z}_n$. Since $p_1^{f(p_1)}$ and $p_2^{f(p_2)}$ are mutually prime when $p_1, p_2 \in P_F$, $p_1 \neq p_2$, we get $\pi_{2k+1}(\mathrm{Aut}(B)) \simeq \bigoplus_{p \in P_F} \mathbb{Z}_{p^{f(p)}}$, completing the proof in this case.

Assume that $\# P_F = \infty$. By Lemma 1.12 we can assume that

$$B = M_{q_1} \otimes M_{l_1} \otimes M_{q_2} \otimes M_{l_2} \otimes \cdots,$$

where $q_i$ and $l_i$ are integers with properties stated in that lemma. We can define a homomorphism

$$\psi\colon \bigtimes_{i \in \mathbb{N}} \pi_{2k+1}(U(q_i)) \to \pi_{2k+1}(\mathrm{Aut}(B))$$

by $\psi([\alpha^i]_{i=1}^{\infty}) = [\mathrm{Ad}\,\alpha^1 \otimes \mathrm{id} \otimes \mathrm{Ad}\,\alpha^2 \otimes \mathrm{id} \otimes \mathrm{Ad}\,\alpha^3 \otimes \cdots]$. By Lemma 1.12, $\psi$ is a surjection. Choose isomorphisms

$$\varphi_i\colon \pi_{2k+1}(U(q_i)) \to \mathbb{Z}.$$

*Claim.* $\psi([\alpha^i]_{i=1}^{\infty})=0 \Leftrightarrow \varphi_i([\alpha^i])\in q_i\mathbb{Z}$, $i\in\mathbb{N}$.

To prove the claim assume first that

$$\varphi_i([\alpha^i])\in q_i\mathbb{Z}, \qquad i\in\mathbb{N}.$$

By Corollary 1.6 there is a $(2k+1)$-loop $\beta$ in $U(q_2)$ such that $[\alpha^1\otimes\mathbb{1}\otimes\mathbb{1}]=[\mathbb{1}\otimes\mathbb{1}\otimes\beta]$ in $\pi_{2k+1}(U(q_1l_1q_2))$. Corollary 1.6 also gives that $\varphi_2([\beta])\in q_2\mathbb{Z}$. Applying Corollary 1.6 inductively in this way we come in a position where we can conclude from Lemma 1.11 that

$$[\operatorname{Ad}\alpha^1\otimes\mathrm{id}\otimes\mathrm{id}\otimes\cdots]=0 \qquad \text{in} \quad \pi_{2k+1}(\operatorname{Aut}(B)).$$

Hence $\psi([\alpha^i]_{i=1}^{\infty})=[\mathrm{id}\otimes\mathrm{id}\otimes\operatorname{Ad}\alpha^2\otimes\mathrm{id}\otimes\operatorname{Ad}\alpha^3\otimes\mathrm{id}\otimes\cdots]$. Applying the preceding argument inductively, we can proceed as in the proof of Lemma 1.11 and obtain a homotopy connecting

$$\operatorname{Ad}\alpha^1\otimes\mathrm{id}\otimes\operatorname{Ad}\alpha^2\otimes\mathrm{id}\otimes\operatorname{Ad}\alpha^3\otimes\mathrm{id}\otimes\cdots$$

to the trivial $(2k+1)$-loop in $\operatorname{Aut}(B)$. Hence $\psi([\alpha^i]_{i=1}^{\infty})=0$. Conversely, assume that this holds. Then

$$[\operatorname{Ad}\alpha^1\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\cdots]=-[\mathrm{id}\otimes\mathrm{id}\otimes\operatorname{Ad}\alpha^2\otimes\mathrm{id}\otimes\operatorname{Ad}\alpha^3\otimes\mathrm{id}\cdots]$$

in $\pi_{2k+1}(\operatorname{Aut}(B))$.

By Lemma 1.14 this implies that there exist $m\in\mathbb{N}$ and a $(2k+1)$-loop $\beta$ in $U(l_1q_2l_2q_3\cdots q_m)$ such that

$$[\alpha^1\otimes\mathbb{1}]=[\mathbb{1}\otimes\beta] \qquad \text{in} \quad \pi_{2k+1}(U(q_1l_1q_2\cdots q_m)).$$

By Corollary 1.7 this implies that $\varphi_1([\alpha^1])\in q_1\mathbb{Z}$. But then the above argument gives that

$$[\operatorname{Ad}\alpha^1\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\cdots]=0 \qquad \text{in} \quad \pi_{2k+1}(\operatorname{Aut}(B)).$$

Hence $[\mathrm{id}\otimes\mathrm{id}\otimes\operatorname{Ad}\alpha^2\otimes\mathrm{id}\otimes\operatorname{Ad}\alpha^3\otimes\mathrm{id}\otimes\cdots]=0$ in $\pi_{2k+1}(\operatorname{Aut}(B))$. Almost a repetition of this argument proves the following statement by induction:

For each $n\in\mathbb{N}$,

$$[\mathrm{id}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\operatorname{Ad}\alpha^n\otimes\mathrm{id}\otimes\operatorname{Ad}\alpha^{n+1}\otimes\mathrm{id}\otimes\operatorname{Ad}\alpha^{n+2}\otimes\cdots]=0$$

in $\pi_{2k+1}(\operatorname{Aut}(B))$ and $\varphi_i([\alpha^i])\in q_i\mathbb{Z}$, $i\leqslant n-1$.

Especially, we see that $\varphi_i([\alpha^i])\in q_i\mathbb{Z}$ for all $i\in\mathbb{N}$.

Thus we have established the claim. As a consequence

$$\pi_{2k+1}(\mathrm{Aut}(B)) \simeq \bigtimes_{i=1}^{\infty} \mathbb{Z}q_i$$

But $\mathbb{Z}_{q_i} \simeq \bigoplus_{p \in N_i} \mathbb{Z}_{p^{f(p)}}$, so we have proved the theorem also in this case. ∎

*Remark* 1.16. For UHF-algebras $A$ with $P_F = \emptyset$, Theorem 1.15 says that all homotopy groups of $\mathrm{Aut}(A)$ are trivial. In this case, $\mathrm{Aut}(A)$ is even contractible. To see this, assume, for the simplicity of exposition, that $A = M_p \otimes M_p \otimes M_p \otimes \cdots$, $p \in P$. Let $\mathrm{Aut}_1(A) = \{\alpha \in \mathrm{Aut}(A) | \alpha|_{A_1} = \mathrm{id}|_{A_1}\}$.

We can define an isomorphism

$$\Phi\colon \quad \mathrm{Aut}(A) \to \mathrm{Aut}_1(A)$$

by

$$\Phi(\alpha)(a_1 \otimes a_2 \otimes a_3 \otimes \cdots) = a_1 \otimes \alpha(a_2 \otimes a_3 \otimes \cdots),$$

where $\{a_i\}_{i=1}^{\infty} \subseteq M_p$ is a sequence such that $a_i = 1$ eventually.

Assume that we can construct a continuous map $F\colon \mathrm{Aut}(A) \times [0, 1] \to \mathrm{Aut}(A)$ such that

$$F(\alpha, 0) = \alpha, \qquad F(\alpha, 1) = \Phi(\alpha), \qquad \alpha \in \mathrm{Aut}(A).$$

It is clear that we can use the fact that $A \cong A \cap A_n'$, $n \in \mathbb{N}$, and the procedure of taking a sequence strictly increasing to 1 to obtain a contraction of $\mathrm{Aut}(A)$ to id.

Hence the problem is to construct $F$. Define a sequence of elements $\theta_n \in \mathrm{Aut}(A)$ by

$$\theta_n(a_1 \otimes a_2 \otimes a_3 \otimes \cdots) = a_n \otimes a_1 \otimes a_2 \otimes a_3 \otimes \cdots,$$

where $\{a_i\}_{i=1}^{\infty} \subseteq M_p$ is a sequence such that $a_i = 1$ eventually.

Let $0 = \delta_0 < \delta_1 < \delta_2 < \cdots$, such that $\lim_n \delta_n = 1$. Since $\theta_{n+1}^{-1}\theta_n$ acts trivially on $A_{n-1}$, $n \geqq 2$, we can find curves

$$G_n\colon \quad [\delta_{n-1}, \delta_n] \to \mathrm{Aut}(A)$$

such that

$$G_n(\delta_{n-1}) = \theta_{n+1}^{-1}\theta_n,$$
$$G_n(\delta_n) = \mathrm{id}, \qquad n \geqq 1,$$

and

$$G_n(t)|_{A_{n-1}} = \mathrm{id}|_{A_{n-1}}, \qquad t \in [\delta_{n-1}, \delta_n], n \geqq 2.$$

Define $G$: $\mathrm{Aut}_1(A)\times[0,1]\to\mathrm{Aut}(A)$ by

$$G(\alpha,t)=\begin{cases}G_n(t)^{-1}\theta_{n+1}^{-1}\alpha\theta_{n+1}G_n(t), & t\in[\delta_{n-1},\delta_n]\\ \Phi^{-1}(\alpha), & t=1.\end{cases}$$

We claim that $G$ is continuous. In order to check this, it suffices to take converging sequences, $\alpha_n\to\alpha$ in $\mathrm{Aut}_1(A)$, $t_n\to 1$ in $[0,1]$, and

$$a=a_1\otimes a_2\otimes\cdots\otimes a_k\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\cdots,$$

in $A$ and show that

$$G(\alpha_n,t_n)(a)\to\Phi^{-1}(\alpha)(a).$$

Choose $\{k_n\}_{n=1}^{\infty}\subseteq\mathbb{N}$ such that $t_n\in[\delta_{k_n-1},\delta_{k_n}]$ and observe that

$$\theta_{k_n+1}G_{k_n}(t)(a)=\mathbb{1}\otimes a_1\otimes a_2\otimes\cdots\otimes a_k\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\cdots=b$$

when $k_n-1\geqq k$. Hence

$$\begin{aligned}\|G(\alpha_n,t_n)(a)-\Phi^{-1}(\alpha)(a)\| &\leqq\|G(\alpha_n,t_n)(a)-G(\alpha,t_n)(a)\|\\ &\quad+\|G(\alpha,t_n)(a)-\Phi^{-1}(\alpha)(a)\|\\ &=\|G_{k_n}(t_n)^{-1}\theta_{k_n+1}^{-1}\alpha_n(b)-G_{k_n}(t_n)^{-1}\theta_{k_n+1}^{-1}\alpha(b)\|\\ &\quad+\|G_{k_n}(t_n)^{-1}\theta_{k_n+1}^{-1}\alpha(b)-\Phi^{-1}(\alpha)(a)\|\\ &\leqq\|\alpha_n(b)-\alpha(b)\|\\ &\quad+\|G_{k_n}(t_n)^{-1}\theta_{k_n+1}^{-1}\alpha(b)-\Phi^{-1}(\alpha)(a)\|\end{aligned}$$

when $k_n-1\geqq k$. Since $G_{k_n}(t_n)^{-1}\to\mathrm{id}$, it suffices to observe that

$$\lim_n\theta_n^{-1}\alpha(b)=\Phi^{-1}(\alpha)(a)$$

which is clear since

$$\alpha(b)=\mathbb{1}\otimes\Phi^{-1}(\alpha)(a_1\otimes a_2\otimes\cdots\otimes a_k\otimes\mathbb{1}\otimes\mathbb{1}\otimes\cdots).$$

We can then define $F$ by

$$F(\alpha,t)=G(\Phi(\alpha),1-t),\qquad\alpha\in\mathrm{Aut}(A),\quad t\in[0,1].$$

## 2

Next we turn to the homotopy groups of $\mathrm{Aut}_u(B)$. Lance proved in [6] that the connected component of the identity in $\mathrm{Aut}_u(B)$ consists of the inner *-automorphisms. Hence $\pi_0(\mathrm{Aut}_u(B))$ is the group of outer automorphisms of $B$.

To calculate the higher homotopy groups of $\mathrm{Aut}_u(B)$, the literature is much more helpful than in the case of $\mathrm{Aut}(B)$. First, we have the following consequence of the Bartle–Graves selection theorem:

PROPOSITION 2.1 (Philips–Raeburn [7]). *Let $A$ be a unital $C^*$-algebra such that every *-derivation of $A$ is inner. Then there is a continuous map*

$$\gamma\colon \{\alpha \in \mathrm{Aut}_u(A) \mid \|\alpha - \mathrm{id}\| < \sqrt{3}\} \to U(A)$$

*such that $\beta = \mathrm{Ad}\,\gamma(\beta)$, $\beta \in \{\alpha \in \mathrm{Aut}_u(A) \mid \|\alpha - \mathrm{id}\| < \sqrt{3}\}$, and $\gamma(\mathrm{id}) = \mathbb{1}$.* ∎

LEMMA 2.2. *Let $\alpha$ be a 1-loop in $\mathrm{Aut}_u(B)$. Then there is a 1-loop $U$ in $U(B)$ such that*

$$\alpha = \mathrm{Ad}\, U.$$

*Proof.* We consider $\alpha$ as a map $\alpha\colon [0, 1] \to \mathrm{Aut}_u(B)$ such that $\alpha_0 = \alpha_1 = \mathrm{id}$.

Choose $N \in \mathbb{N}$ such that $|t_1 - t_2| \leqslant 1/N$ implies that

$$\|\alpha_{t_1} - \alpha_{t_2}\| < \sqrt{3}.$$

Since every *-derivation of a simple $C^*$-algebra is inner [8, 8.6.10], we can apply Proposition 2.1 and define

$$U_t^i = \gamma(\alpha_t \alpha_{i/N}^{-1}), \qquad t \in \left[\frac{i}{N}, \frac{i+1}{N}\right], \quad i = 0, 1, 2, \ldots, N-1.$$

Then $\mathrm{Ad}\, U_t^i = \alpha_t \alpha_{i/N}^{-1}$, $t \in [i/N, (i+1)/N]$ and

$$U_{i/N}^i = \mathbb{1}, \qquad i = 0, 1, 2, \ldots, N-1.$$

Define

$$V_t = \begin{cases} U_t^0, & t \in \left[0, \frac{1}{N}\right] \\ U_t^i U_{i/N}^{i-1} \cdots U_{1/N}^0, & t \in \left[\frac{i}{N}, \frac{i+1}{N}\right], \quad 1 \leqslant i \leqslant N-1. \end{cases}$$

Then $V: [0, 1] \to U(A)$ is continuous, $V_0 = \mathbb{1}$ and $\alpha_t = \operatorname{Ad} V_t$, $t \in [0, 1]$. Especially, $V_1 = \lambda \mathbb{1}$ for some $\lambda \in S^1$. Let $f: [0, 1] \to S^1$ be a continuous map such that

$$f(0) = 1, \qquad f(1) = \lambda^{-1}.$$

Define $U_t = f(t) V_t$, $t \in [0, 1]$. Then $U$ is a 1-loop with the right property. ∎

LEMMA 2.3. *Let $B = M_{p_1} \otimes M_{p_2} \otimes \cdots = \bigotimes_{i=1}^{\infty} M_{p_i}$, and let $d_n = \prod_{i=1}^{n} p_i$. Then the maps $U(d_n) \ni U \to U \otimes \mathbb{1} \in U(d_{n+1})$ induce homomorphisms $\pi_k(U(d_n)) \to \pi_k(U(d_{n+1}))$ for all $k \in \mathbb{N}$, and we have an isomorphism*

$$\pi_k(\operatorname{Aut}_u(B)) \simeq \varinjlim_n \pi_k(U(d_n))$$

*for $k \geqslant 3$.*

*Proof.* First note that by the result of Lance cited above

$$\pi_k(\operatorname{Aut}_u(B)) \simeq \pi_k(\operatorname{Inn}(B)), \qquad k \geqslant 1,$$

where $\operatorname{Inn}(B)$ denotes the group of inner *-automorphisms of $B$ endowed with the uniform norm topology.

Proposition 2.1 implies that

$$U(B) \xrightarrow{\operatorname{Ad}} \operatorname{Inn}(B)$$

is a fibre bundle with fibre $S^1$. Hence $\pi_k(\operatorname{Inn}(B)) \simeq \pi_k(U(B), S^1)$ for $k \geqslant 2$, by [9, 17.1].

Since $\pi_k(S^1) = 0$ for $k \geqslant 2$, the homotopy sequence for the pair $(U(B), S^1)$ yields that

$$\pi_k(\operatorname{Aut}_u(B)) \simeq \pi_k(U(B)) \qquad \text{for} \quad k \geqslant 3$$

The conclusion then follows from Lemma 1.1. ∎

LEMMA 2.4. *Adopt the notation of Lemma 2.3. The mappings $\pi_1(U(d_n)) \to \pi_1(U(d_{n+1}))$ induce homomorphisms*

$$\pi_1(PU(d_n)) \to \pi_1(PU(d_{n+1}))$$

*(where $PU(m)$ denotes the projective unitary group $U(m)/S^1$) and we have an isomorphism*

$$\varinjlim_n \pi_1(PU(d_n)) \simeq \pi_1(\operatorname{Aut}_u(B)).$$

*Proof.* We define $\tau_n \colon U(d_n) \to \mathrm{Aut}_u(B)$ by

$$\tau_n(u) = \mathrm{Ad}\, u \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots.$$

The $\tau_n$'s induce homomorphisms

$$\tau_n \colon PU(d_n) \to \mathrm{Aut}_u(B)$$

such that the induced maps

$$\pi_1(PU(d_n)) \to \pi_1(\mathrm{Aut}_u(B))$$

are compatible with the homomorphisms defining the direct limit $\varinjlim_n \pi_1(PU(d_n))$.

Hence we obtain a homomorphism

$$\tau \colon \varinjlim \pi_1(PU(d_n)) \to \pi_1(\mathrm{Aut}_u(B)).$$

Lemmas 2.2 and 1.1 give that $\tau$ is surjective.

To prove that $\tau$ is injective let $u$ be a 1-loop in $U(d_n)$ and assume that

$$[\mathrm{Ad}\, u \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots] = 0 \qquad \text{in} \quad \pi_1(\mathrm{Aut}_u(B)).$$

Let $V = u \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots$, and let

$$F \colon [0, 1] \times [0, 1] \to \mathrm{Aut}_u(B)$$

be continuous such that

$$\begin{aligned} &G(0, s) = \mathrm{Ad}\, V_s, && s \in [0, 1], \\ &F(1, s) = \mathrm{id}, && \\ &F(t, 0) = F(t, 1) = \mathrm{id}, && t \in [0, 1]. \end{aligned}$$

Choose $N \in \mathbb{N}$ such that

$$|t_1 - t_2| \leqslant 1/N \Rightarrow \sup\{\|F(t_1, s) - F(t_2, s)\| \mid s \in [0, 1]\} < \sqrt{3}.$$

Let $U^i_{t,s} = \gamma(F(t, s)\, F(i/N, s)^{-1})$, $s \in [0, 1]$, $t \in [i/N, (i+1)/N]$ $i = 0, 1, 2, \ldots, N-1$. Then $U^i_{t,0} = \mathbb{1} = U^i_{t,1}$ for all $t \in [i/N, (i+1)/N]$, $i = 0, 1, 2, \ldots, N-1$ and

$$\mathrm{Ad}\, U^i_{t,s} = F(t, s)\, F\left(\frac{i}{N}, s\right)^{-1}, \qquad t \in \left[\frac{i}{N}, \frac{i+1}{N}\right], s \in [0, 1],$$

$i = 0, 1, 2, \ldots, N-1$.

Also $U^i_{i/N,s} = \mathbb{1}$ for all $s \in [0, 1]$. Define

$$W_{t,s} = \begin{cases} U^0_{t,s} V_s, & t \in \left[0, \frac{1}{N}\right], \quad s \in [0, 1] \\ U^i_{t,s} U^{i-1}_{i/N,s} \cdots U^0_{i/N,s} V_s, & t \in \left[\frac{i}{N}, \frac{i+1}{N}\right], \quad s \in [0, 1], 1 \leqslant i \leqslant N-1. \end{cases}$$

Then $W$: $[0, 1] \times [0, 1] \to U(B)$ is continuous,

$$\mathrm{Ad}\, W_{t,s} = F(t, s) \qquad \text{for all } (t, s) \in [0, 1]^2$$

and $W_{t,0} = W_{t,1} = \mathbb{1}$ for all $t \in [0, 1]$. Especially, $W_{1,\cdot}$ is a 1-loop in $S^1\mathbb{1}$. Thus $W$ is a homotopy in $U(B)$ connecting the 1-loop $V$ in $U(B)$ to a 1-loop in $S^1\mathbb{1}$. Since $2d_n - 1 \geqslant 1$ for all $n$, Lemma 1.1 and Corollary 1.5 together yield that the induced map

$$\pi_1(U(d_n)) \to \pi_1(U(B))$$

is injective.

Hence we conclude that $u$ is homotopic to a 1-loop in $S^1\mathbb{1} \subseteq U(d_n)$. Thus $u$ represents the trivial element in $\pi_1(PU(d_n))$. This shows that $\tau$ is injective and ends the proof. ∎

LEMMA 2.5. $\pi_2(\mathrm{Aut}_u(B)) = 0$.

*Proof.* From the proof of Lemma 2.3, we recall that

$$\pi_2(\mathrm{Aut}_u(B)) \simeq \pi_2(U(B), S^1).$$

From the homotopy sequence of $(U(B), S^1)$ we get the exact piece

$$0 \to \pi_2(U(B), S^1) \to \pi_1(S^1) \to \pi_1(U(B)),$$

where we have used that $\pi_2(U(B)) = \lim_n \pi_2(U(d_n)) = 0$ by Lemma 1.1 and 1.2. But the last map $\pi_1(S^1) \to \pi_1(U(B))$ is injective, so $\pi_2(U(B), S^1) = 0$. ∎

THEOREM 2.6. *Let $B$ be an infinite-dimensional* UHF-*algebra $B = M_{p_1} \otimes M_{p_2} \otimes M_{p_3} \otimes \cdots = \bigotimes_{i=1}^{\infty} M_{p_i}$. Then*

$$\pi_{2k}(\mathrm{Aut}_u(B)) = 0, \qquad k = 1, 2, 3, \ldots,$$

$$\pi_{2k+1}(\mathrm{Aut}_u(B)) \simeq K_0(B), \qquad k = 1, 2, 3, \ldots,$$

*and*

$$\pi_1(\mathrm{Aut}_u(B)) \simeq \varinjlim_k \mathbb{Z}_{d_k},$$

*where* $d_k = \prod_{i=1}^k p_i$ *and* $\mathbb{Z}_{d_k} \to \mathbb{Z}_{d_{k+1}}$ *is induced by multiplication by* $p_{k+1}$.

*Proof.* Since $\pi_{2k}(\mathrm{Aut}_u(B)) \simeq \varinjlim \pi_{2k}(U(d_n))$ for $k \geqslant 2$, by Lemma 2.3, we conclude that $\pi_{2k}(\mathrm{Aut}_u(B)) = 0$ because $\pi_{2k}(U(d_n)) = 0$ when $2d_n - 1 \geqslant 2k$. That $\pi_2(\mathrm{Aut}_u(B)) = 0$ is Lemma 2.5. Since $\pi_1(PU(d_n)) \simeq \mathbb{Z}_{d_n}$ for all $n \in \mathbb{N}$, the last statement follows from Lemma 2.4. Fix $k$ odd, $k \geqslant 3$. When $2d_m - 1 \geqslant k$, Lemma 1.3 gives us an infinite commutative diagram

$$\begin{tikzcd}
\pi_k(U(d_m)) \arrow[r] \arrow[d, no head] & \pi_k(U(d_{m+1})) \arrow[r] \arrow[d, no head] & \pi_k(U(d_{m+2})) \arrow[r] \arrow[d, no head] & \cdots \\
\mathbb{Z} \arrow[r, "z \to p_{m+1} z"] & \mathbb{Z} \arrow[r, "z \to p_{m+2} z"] & \mathbb{Z} \arrow[r] & {}
\end{tikzcd}$$

where the vertical lines indicate isomorphisms. The direct limit group of the lower system is $K_0(B)$, so

$$\pi_k(\mathrm{Aut}_u(B)) \simeq K_0(B)$$

by Lemma 2.3. ∎

It should be noted that also $\pi_1(\mathrm{Aut}_u(B))$ has a description in terms of $K_0(B)$. If we use the unique positive normalized trace on $B$ to identify $K_0(B)$ as a subgroup of the rationals, then

$$\pi_1(\mathrm{Aut}_u(B)) \simeq K_0(B)/\mathbb{Z}.$$

According to [4, Theorem 4.6] we have

$$K_0(B) \simeq \pi_1(U(B)).$$

Hence $\pi_{2k+1}(\mathrm{Aut}_u(B)) \simeq \pi_1(U(B))$ for $k \in \mathbb{N}$.

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